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BASIC COMPUTATIONAL PROBLEMS IN THE FINITE ELEMENT ANALYSIS OF SHELLS†

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Abstract—The purpose of this paper is to examine the two fundamental questions in the finite element analysis of shells; namely, the rates of convergence resulting from different interpolation schemes for the inplane and normal displacements, and the dependence of the condition number of the resulting algebraic system on the various parameters both of the shell and of the discretization.

1. INTRODUCTION

THERE is a fundamental difference between the plate and the shell. Only in the latter there exists a coupling between the tangential (in-plane) and normal displacements. It has been shown [1] that for the plate (or equivalently the beam), the condition number of the global matrix is independent of the intrinsic parameters of the plate (such as thickness to length ratio), and that it varies as N_{es}^4, where N_{es} denotes the number of elements per side. For a deep shell, this coupling may completely alter the behavior of the condition number as a function of both the number of elements and the shell's intrinsic parameters.

In high order problems where the condition number grows rapidly with the number of elements, and in other problems, as with certain shells, where it may critically increase with certain parameters, round-off errors become a serious problem and may even dominate the total error in the numerical solution. In such cases no effort should be spared in searching for techniques for neutralizing the effect of ill-conditioning.

Two principal avenues are open for improving the ratio between discretization and round-off errors: the first involves transformation of the variables, and the second, the use of higher order elements. Both these possibilities are explored here.

The coupling between the tangential and normal displacements also gives rise to questions concerning their mode of interpolation. The normal and tangential displacements can either be interpolated independently [2], which is the more usual technique, or in a coupled manner [3]. If they are interpolated separately, which is technically more convenient, then the question arises of what the relation is between the order of the interpolating polynomials for the tangential and normal displacements such that maximal efficiency can be attained. Thus, a given rate of convergence is attained with a minimal number of degrees of freedom per element. Numerical experiments [4] indicate that optimal efficiency is often obtained with equal interpolation schemes for both the inplane and normal displacements. This is confirmed here theoretically.

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2. CONDITION NUMBER

The condition number $C_n(K)$ with respect to the Euclidean norm of the positive definite matrix K is defined as the ratio between the maximal (Nth) eigenvalue of K, λ_N^K and the minimal (1st) eigenvalue λ_1^K of K. This number provides a measure of the computational effort required for obtaining a certain accuracy in solving the linear system $K_x = b$.

The condition number depends basically on two aspects of the problem: on the nature of the continuous equations and on the discretization method. The conditioning associated with the first aspect is referred to as natural. The conditioning depending on the types of elements employed and on their geometrical parameters is referred to as triangularization conditioning. Of primary interest here is the natural condition number intimately associated with the nature of the shell problem. Methods for evaluating the effects of the triangularization have been discussed elsewhere [1].

Let us denote by K and M the stiffness and mass matrices respectively, and by λ_1^K, λ_N^K, λ_1^M and λ_N^M their extremal eigenvalues. From Rayleigh's principle

$$\frac{\lambda_N}{\lambda_1}\frac{\lambda_1^M}{\lambda_N^M} \leq C_n(K) \leq \frac{\lambda_N}{\lambda_1}\frac{\lambda_N^M}{\lambda_1^M} \tag{1}$$

is readily obtained where λ_1 and λ_N are the minimal and maximal eigenvalues of

$$K_{xn} - \lambda_n M_{xn} = 0, \qquad n = 1, 2, \ldots, N. \tag{2}$$

In the case of discretization by a regular mesh of finite elements, M is bounded from below and above, and hence the ratio λ_N/λ_1 provides, in this case, a good estimate for $C_n(K)$. Moreover, since the error in λ_N is of the order 1 and in λ_1 much smaller, for the present purposes the approximate eigenvalues in equation (1) can be replaced by the exact ones, thereby revealing the dependence of the natural condition number on the nature of the continuous problem.

In the more general case of irregular meshes the bound [1]

$$C_n(K) \leq \frac{p_{\max}}{\lambda_1}\frac{\max_e(\lambda_n^{k^e})}{\min_e(\lambda_1^{m^e})} \tag{3}$$

can be used, where $p_{\max}$ is the maximal number of elements meeting at a nodal point and $\lambda_n^{k^e}$ and $\lambda_1^{m^e}$ the maximal (nth) and minimal (1st) eigenvalues of the element matrices k_e and m_e corresponding to K and M, respectively. $\mathrm{Max}_e(\lambda)$ denotes the maximal value of λ over $e = 1, 2, \ldots$.

To explore the dependence of $C_n(K)$ on the basic shell parameters the bound given by equation (3) is used on the simple one-dimensional case of a circular arch. The element matrices k_e and m_e are derived in this case from the following energy expressions

$$\mathscr{E}_p = \frac{1}{2}Et\int_{-h}^{h}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}s}+\frac{w}{r}\right)^2 + \frac{t^2}{12}\left(\frac{\mathrm{d}u}{r\,\mathrm{d}s}-\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}\right)^2\right]\mathrm{d}s \tag{4}$$

$$\mathscr{E}_k = \frac{1}{2}\int_{-h}^{h}(u^2+w^2)\,\mathrm{d}s \tag{5}$$

in which h is half the element's size, t the thickness of the shell and r the radius of curvature. Transforming the element from the s-system to the ξ-system where it is bounded by $-1 \leq \xi \leq 1$, equations (4) and (5) become

$$\mathscr{E}_p = \frac{Et}{2h}\int_{-1}^{1}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}+\frac{h}{r}w\right)^2+\frac{1}{12}\frac{t^2}{h^2}\left(\frac{h}{r}\frac{\mathrm{d}u}{\mathrm{d}\xi}-\frac{\mathrm{d}^2w}{\mathrm{d}\xi^2}\right)^2\right]\mathrm{d}\xi \tag{6}$$

$$\mathscr{E}_k = \frac{1}{2}h\int_{-1}^{1}(u^2+w^2)\,\mathrm{d}\xi. \tag{7}$$

Also

$$\frac{h}{r}=\frac{\beta}{2N_e}, \qquad \frac{t}{h}=2N_e\tau \tag{8}$$

where β is the opening angle of the arch, L the length of the arch ($L = \beta r$), $\tau = t/L$ and N_e the number of finite elements in the discretization.

No loss of generality will occur in equations (6) and (7) if the following conditions are set: $h = 1$ and $Et/h = 1$. The element stiffness and mass matrices derived from equations (6) and (7) are denoted by k_e and m_e, respectively. $\lambda_n^{k_e}$ denotes the maximal eigenvalue of k_e, and $\lambda_1^{m_e}$ and $\lambda_n^{m_e}$ the minimal and maximal eigenvalues, respectively, of m_e.

The first element on which analysis will now be carried out is denoted by $B(2, 3)$, which indicates that within the element u is interpolated by quadratic polynomials, and w by cubic. This element is associated with the 7 nodal variables $(u_1, w_1, w_{\xi 1}, u_2, u_3, w_3, w_{\xi 3})$, but the slope variables can be derived either with respect to ξ or with respect to $s = h\xi$. Subsequently a search will be made for an h which yields the best conditioned matrix.

Since $p_{\max}$ and $\lambda_1^{m_e}$ are constants, the bound (3) on $C_n(K)$ depends only on λ_1 and $\lambda_n^{k_e}$. λ_1 itself depends on the parameters of the arch and on the boundary conditions, and it can generally be written as [5]

$$\lambda_1 = c\tau^2/(4N_e^2) \tag{9}$$

where c depends on β, τ and the boundary conditions. For the case of a simply supported arch with $\tau < 1/50$ and $\beta > \pi/5$, c is almost constant.

Figure 1 depicts the variation of λ_7^k (since all k_e are equal the subscript e is dropped from k_e) for different values of L/t vs. N_e. It has been drawn only for $\beta = \pi/2$ since there are only slight variations of λ_7^k with β. It can be seen from Fig. 1 that in the range $N_e < L/t$, λ_7^k is nearly constant, and hence in this range $C_n(K)$ becomes

$$C_n(K) = c\left(\frac{L}{t}\right)^2 N_e^2 \tag{10}$$

where c denotes a numerical coefficient.

For the range $N_e > L/t$ it becomes

$$C_n(K) = cN_e^4. \tag{11}$$

The range $N_e > L/t$ is of little interest, however, as the size of the element there is comparable to the thickness. Useful results are expected with far fewer elements.

In Figs. 2 and 3 the predictions of equation (10) are well confirmed numerically. This behavior of the natural condition number can also be explained by equation (1), by

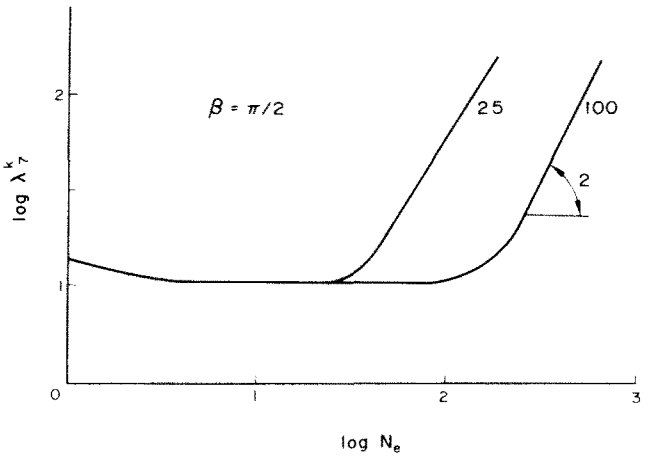

FIG. 1. Variation of λ_7^k for a circular element with the number of elements N_e. Numbers on the curves indicate values of the ratio L/t.

examining the dynamical behavior of the shell. Consider, for example, a complete ring. Its bending or flexural frequencies (in fact, their square) are given by [6]

$$\lambda_n^{(b)} = \frac{\pi t^3}{4r^4} \frac{n^2(n^2-1)^2}{n^2+1}, \qquad n = 2, 3, \ldots \tag{12}$$

while the extensional frequencies are given by

$$\lambda_n^{(l)} = \frac{\pi t}{r^2}(1+n^2), \qquad n = 0, 1. \tag{13}$$

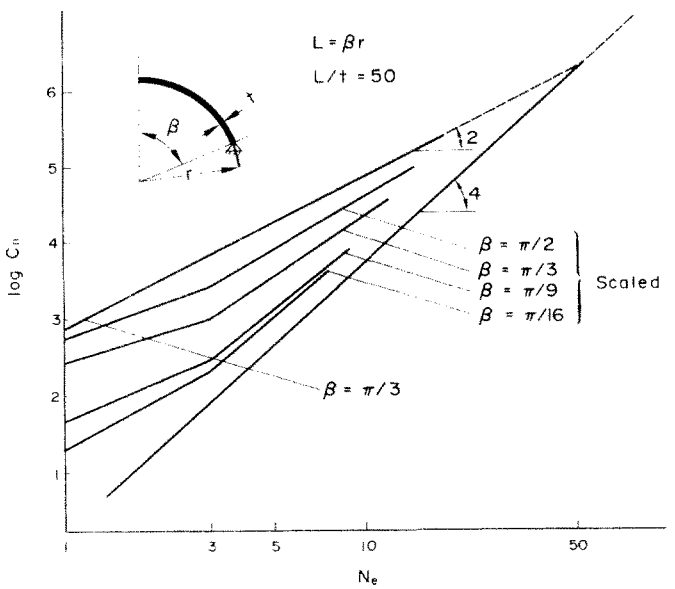

FIG. 2. Variation of the condition number C_n with N_e, in the case of simply supported circular arch with and without energy scaling.

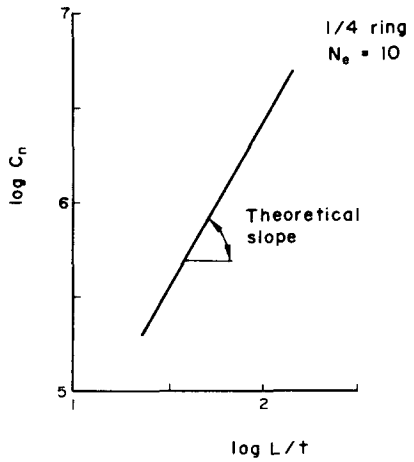

FIG. 3. Variation of the condition number C_n, in the case of a 1/4 ring, as a function of the length to thickness ratio.

The lowest eigenvalue is evidently flexural and has the value

$$\lambda_2^{(b)} = \frac{9}{5}\frac{\pi t^3}{r^4}. \tag{14}$$

For $n < 2r/t$ the maximal eigenvalue is extensional, so that $C_n(K) = cn^2r^2/t^2$ and n, the mode number, is proportional to N_e. For $h > 2r/t$ the maximal eigenvalue is flexural and $C_n(K) = ch^4$.

From the above argument it can be concluded that the natural condition number should not depend strongly on the type of element employed in the discretization process [the influence of the type of element will be manifested in λ_N^M/λ_1^M in equation (1)], but rather on the type of problem. Thus by using higher order elements an increase is expected in the rate of convergence without much alteration of the condition number. This in turn permits a certain control over the ratio between the discretization and round-off errors. This possibility will be examined in the next section.

In two-dimensional cases the maximal frequency depends on the number of waves per side, and hence the condition number also will grow as a function of the *number of elements per side* rather than of the total number of elements in the domain. This explains the advantage of experimenting with one-dimensional problems where ill-conditioning can be achieved with fewer variables (in fact square root) than in two-dimensional problems.

Now the possibility of improving $C_n(K)$ by a transformation of variables (scaling) is examined. In fact, as β is decreased, the bending and extensional energies in equation (6) tend to decouple, and at the limit, $\beta = 0$, the condition number [as given by equation (10)] loses significance. It is proposed, therefore, to scale the extensional energy, as for plate problems, via the variable transformation

$$u = \hat{u}\frac{t}{h} \tag{15}$$

which transforms equations (6) and (7) into

$$\hat{\mathscr{E}}_p = \frac{1}{2}E\frac{t^3}{h^3}\int_{-1}^{1}\left[\left(\frac{d\hat{u}}{d\xi}+\frac{\beta h^2}{Lt}w\right)^2+\frac{1}{12}\left(\frac{\beta t}{L}\frac{d\hat{u}}{d\xi}-\frac{d^2w}{d\xi^2}\right)^2\right]d\xi \tag{16}$$

and

$$\hat{\mathscr{E}}_k = \frac{1}{2}h\int_{-1}^{1}\left(\hat{u}\frac{t^2}{h^2}+w^2\right)d\xi. \tag{17}$$

The element stiffness and mass matrices derived from equations (16) and (17) are denoted by $\hat{k}$ and $\hat{m}$, respectively. By $\lambda_n^{\hat{k}}$ is denoted the maximal eigenvalue of $\hat{k}$, by $\lambda_1^{\hat{m}}$ the minimal eigenvalue of $\hat{m}$ and by $\lambda_n^{\hat{m}}$ the maximal eigenvalue of $\hat{m}$. For the 7 d.o.f. element $n = 7$.

Also

$$\frac{\beta h^2}{tL} = \beta\frac{L}{t}\frac{1}{4N_e^2}. \tag{18}$$

For $N_e > 5$ and $\beta < \pi/2$, $\lambda_7^{\hat{k}}$ is fairly constant (assuming for the sake of convenience that $Et^3/h^3 = 1$ and $h = 1$), while for moderate values of t/h, $\lambda_1^{\hat{m}}$ is independent of t/h. Thus it is concluded from equation (3) that for $N_e > 5$, $C_n(K) = cN_e^4$ as in the beam problem. As seen in Fig. 2, scaling becomes indeed more effective [or $C_n(K)$ more meaningful] as β approaches zero. For $\beta > \pi/3$ it is only partially useful in improving the condition of K. Nevertheless, in case L/t is large it would be sensible to assume a priori that the bending solution is inextensional.

Apart from energy scaling, the condition of K can be improved by choosing an appropriate scale for h in the nodal variable $(dw/h\,d\xi)_i$. In Fig. 5 $C_n(K)$ for a quarter of a ring is drawn vs. h. It is seen that as in the case of a beam there exists here an optimal h, which is nearly equal to 7, but in the range $1 < h < 10$ the changes in $C_n(K)$ with respect to the scale of h are only slight.

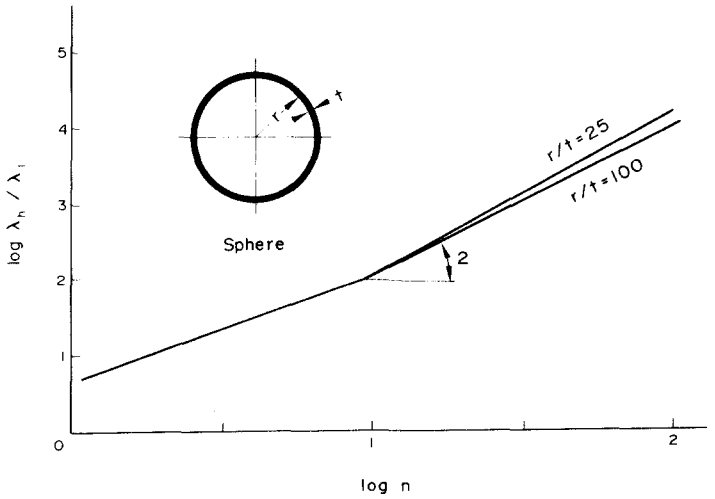

FIG. 4. Variation of the ratios between the first and nth eigenvalues in a spherical shell as a function of n for different radius to thickness ratios.

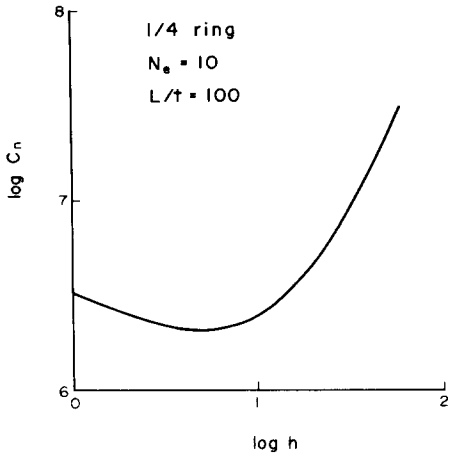

FIG. 5. Variation of C_n, in the case of a 1/4 ring, with h appearing in the slope variable ($dw/h\,d\xi$).

3. DISCRETIZATION ERRORS

Where e_s and e_b denote the true extensional and bending strains, respectively, the variational technique for obtaining approximate solutions consists of looking for an admissible $\tilde{e}_s$ and $\tilde{e}_b$ that will minimize the expression

$$\delta\mathscr{E}_p = \int_0^L \left[(e_s - \tilde{e}_s)^2 + \frac{t^2}{12}(e_b - \tilde{e}_b)^2 \right] ds. \tag{19}$$

Any arbitrary $\tilde{e}_s$ and $\tilde{e}_b$ chosen from an admissible finite element space yields a $\delta\mathscr{E}_p$ higher than that at the approximate solution. In view of this, the technique [8] to use for estimating the rate of convergence is to choose a convenient (but reasonable) $\tilde{e}_s$ and $\tilde{e}_b$ and to estimate with them the error in the energy $\delta\mathscr{E}_p$. This estimate is an upper bound of the error in the approximate solution. A natural choice for $\tilde{e}_s$ and $\tilde{e}_b$ is that obtained from the interpolate of the true solution. In this case, if u is interpolated by polynomials of degree p, and w by polynomials of degree q, then by Taylor's theorem [9]

$$\begin{aligned}
\left| \frac{du}{ds} - \frac{d\tilde{u}}{ds} \right| &\leq c_1 h^p \max \left| \frac{d^{p+1}u}{ds^{p+1}} \right| \\
|w - \tilde{w}| &\leq c_2 h^{q+1} \max \left| \frac{d^{q+1}w}{ds^{q+1}} \right| \\
\left| \frac{d^2w}{ds^2} - \frac{d^2\tilde{w}}{ds^2} \right| &\leq c_3 h^{q-1} \max \left| \frac{d^{q+1}w}{ds^{q+1}} \right|
\end{aligned} \tag{20}$$

where $\max|\Delta|$ indicates the maximal absolute value of Δ in the complete structure. The coefficients c_1, c_2 and c_3 arise from the residual terms in Taylor's expansion, and hence if p and q are not far apart the coefficients also are not far apart.

In order to establish the relation between p and q yielding a comparable error in the extensional and bending energies, it is observed that

$$e_s - \tilde{e}_s = \left(\frac{du}{ds} - \frac{d\tilde{u}}{ds}\right) + \frac{1}{r}(w - \tilde{w})$$
$$e_b - \tilde{e}_b = \left(\frac{du}{ds} - \frac{d\tilde{u}}{ds}\right)\frac{1}{r} + \left(\frac{d^2w}{ds^2} - \frac{d^2\tilde{w}}{ds^2}\right). \tag{21}$$

According to equation (20), the errors in e_s and e_b result principally from the terms du/ds and d^2w/ds^2, respectively. Hence the error in the energies becomes of comparable magnitude under the condition

$$\frac{h^p}{r^{p+1}} \max \frac{d^{p+1}u}{d\theta^{p+1}} = \frac{th^{q-1}}{r^{q+1}} \max \left|\frac{d^{q+1}w}{d\theta^{q+1}}\right| \tag{22}$$

where $s = r\theta$.

For an (almost) inextensional solution $w = du/d\theta$, and thus for a reasonable behavior ($d^pw/d\theta^p$ being comparable to $d^qw/d\theta^q$) of w, equation (22) is reduced to

$$1 = th^{-1-p+q}r^{p-q}. \tag{23}$$

Asymptotically, as $h \to t$, equation (23) predicts that optimal efficiency will be achieved with $p = q$. This has indeed been well confirmed numerically, but in addition it was noted [4] that $p = q-1$ may also result in a satisfactory element.

Another possibility for obtaining efficient interpolation schemes for u and w is by starting with the strains; assume for them a certain variation and integrate for obtaining u and w. For the 7 d.o.f. element $B(2, 3)$ this would amount to starting with

$$\frac{du}{ds} + \frac{w}{r} = a_1 + a_2 s + \varphi_1(s)$$
$$\frac{du}{r\,ds} - \frac{d^2w}{ds^2} = b_1 + b_2 s + \varphi_2(s) \tag{24}$$

where a_1, a_2, b_1 and b_2 are undetermined coefficients, and $\varphi_1(s)$ and $\varphi_2(s)$ include higher order terms of s.

Integration of the system (24) will furnish a coupled interpolation scheme [3] for u and w which includes trigonometric functions. This type of element is also shown [10] to be very efficient, but unfortunately its derivation for the general case is rather cumbersome.

4. HIGHER ORDER ELEMENTS

Still left to be discussed is the effect on the round-off errors of increasing the order of the element. To test this numerically an element is employed which has the 11 d.o.f. (u_1, $u_{\xi 1}$, w_1, $w_{\xi 1}$, $w_{\xi\xi 1}$, u_2, u_3, $w_{\xi 3}$, w_3, $w_{\xi 3}$, $w_{\xi\xi 3}$) and in which w is interpolated by a polynomial of the 5th degree and u by a polynomial of the 4th degree. This element therefore is denoted here as $B(4, 5)$.

In order to test the effect of round-off errors with different elements, the problem of a ring loaded diametrically by concentrated forces is solved twice [in single (24 bits) and double precision], once with $B(2, 3)$ and once with $B(4, 5)$. It is seen from Fig. 6 that even though round-off errors in the $B(4, 5)$ calculations become dominant earlier than in the $B(2, 3)$ calculations, the higher order element permits the achievement of much greater accuracy. Moreover, the higher order element proves to be more efficient. In spite of the

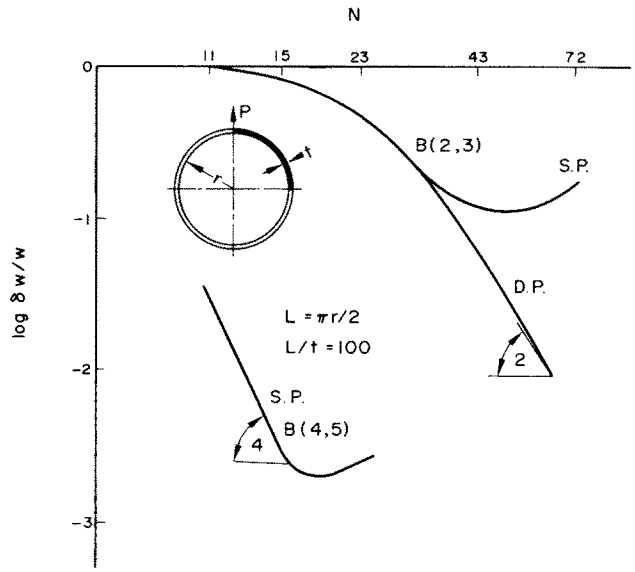

FIG. 6. Convergence of the maximal displacement for ring loaded with opposite forces. Calculation carried out in single (S.P.) and double precision (D.P.) with both the $B(2, 3)$ and $B(4, 5)$ elements. N denotes the total number of variables.

fact that $B(4, 5)$ produces a matrix with a band width of only about twice that produced by $B(2, 3)$, the accuracy obtained with two $B(4, 5)$ elements (17 degrees of freedom) is greater than that achieved with 18 $B(2, 3)$ elements (72 degrees of freedom), using double precision computations. The number of operations required to solve a symmetric linear system with N variables and a band width equal to $2k+1$ is approximately given by Nk^2. Taking into account the fact that double precision operations are about 6 times slower than single precision operations, it is found that, where time is concerned, the higher order element $B(4, 5)$ is about 9 times more efficient than the $B(2, 3)$ element.

5. CONCLUSIONS

For a shell where the flexural and extensional vibration modes can be decoupled, the spectral condition number $C_n(K)$ of the stiffness matrix K generated by a uniform mesh of finite elements is expressed by

$$C_n(K) = c_1\left(\frac{r}{t}\right)^2 N_{es}^2, \qquad N_{es} < r/t \tag{25}$$

$$C_n(K) = c_2 N_{es}^4, \qquad N_{es} > r/t \tag{26}$$

where r/t is the radius of curvature to thickness ratio, N_{es} the number of elements per side, and c_1 and c_2 positive constants. These constants depend only slightly on the order of the element (the order of the interpolation polynomials inside it).

For the case of a 1/4 ring, for instance, $c_1 = 100$, and hence with 10 elements and $r/t = 100$ the condition number reaches 10^6. This is approximately the limit for obtaining significant results on a computer with 24 bits (7·2 digits).

As the shell becomes more and more shallow, scaling via the variable transformation

$$u = \hat{u}\frac{t}{h} \tag{27}$$

tends to eliminate the ratio r/t from expression (25) for the condition number and at the limit $\beta = 0$, $C_n(K) = cN_{es}^4$ as in flat plates.

For a shell which behaves like a membrane because the extensional and flexural modes cannot be decoupled, the condition number grows as N_{es}^2 as in second order problems. (See also Refs. [1, 11].)

As the thickness of the shells is reduced ($t/L \ll 1$), the factor $(t/h)^2$ multiplying the bending energy expression in equation (6) will also be reduced. Since the total energy (stretching plus bending) is minimized for obtaining the finite element solution, the in-plain strains will therefore be forced to diminish also. However, beyond a certain value of t/L its further reduction will have no practical influence on the accuracy. This suggests that for small t/L ratios the bending energy portion can be multiplied by an additional factor δ such that $\delta > 1$. An estimate for the value of this factor is found as follows: the relative contribution of the stretching energy to the displacements is $0(t^2/L^2)$. If the required error in the solution is $10^{-\beta}$, then $\delta t^2/L^2 = 10^{-\beta}$ or $\delta = 10^{-\beta}L^2/t^2$. With these equalities the factor t^2/h^2 in equation (6) can be replaced by either $\delta t^2/h^2$ or $10^{-\beta}$. Of course, this should be done only when $(t/h)^2 < 10^{-\beta}$. In case $(t/h)^2$ is replaced in equation (6) by $10^{-\beta}$, the condition number of the global stiffness matrix will become

$$C_n(K) = c\,10^{\beta}N_{es}^4 \tag{28}$$

and $(L/t)^2$ is removed from the condition number as given in equation (25).

In either case the condition of the stiffness matrix deteriorates rather quickly and this deterioration inevitably results in serious losses in the accuracy of the computed solution.

Since the natural condition number (in the case of uniform meshes) depends primarily on the intrinsic parameters of the shell and not so much on the order of the problem, the relative effect of round-off errors can be made to diminish by increasing the order of the finite elements. In this regard, the low order finite element schemes, like those generated from a "flat plate approximation" to the shell, should be avoided.

The higher order elements require more programming labor, but since this effort is required only once, it is worthwhile.

For obtaining a maximal efficiency in the element (that is, a certain rate of convergence obtained with the minimal number of degrees of freedom) both the in-plane and transverse displacements should be interpolated by polynomials of the same order.

It results from equation (20) that if w and u are interpolated by complete polynomials of degree q and p, respectively, such that $p \geq q$, then the error in the energy is

$$\delta\mathscr{E}_p \leq cN_{es}^{-2(q-1)} \tag{29}$$

Since in the finite element solution the energy is *minimized*, the actual error in the energy will be *at most* that predicted by equation (29).

The situation in eigenproblems is similar to that in static problems.

If in a flat plate ($m = 2$) or in a flat membrane ($m = 1$) the transverse displacement is interpolated inside the element by a complete set of polynomials of degree p, then the discretization error $\delta\lambda_r$ in the rth eigenvalue λ_r is given by [1, 12]

$$\delta\lambda_r/\lambda_r \leq cN_{es}^{-2(p+1-m)}(\lambda_r/\lambda_1)^{(p+1-m)/m} \tag{30}$$

where the constant c is independent of N_{es} and (λ_r/λ_1).

Regarding the round-off errors in eigenproblems, the relative error in the rth eigenvalue due to the machine representation of the stiffness and mass matrices K and M is given by [1]

$$\delta\lambda_r/\lambda_r \leq \lambda_N^K 10^{-s}/\lambda_1^M\lambda_r \tag{31}$$

where s denotes here the number of significant digits in the computer. For the shell problem, if K is normalized such that $\lambda_1^K = 0(1)$, then introduction of $C_n(K)$ of equation (25) into equation (31) results in

$$\frac{\delta\lambda_r}{\lambda_r} \leq c\frac{N_{es}^2(r/t)^2}{\lambda_1^M\lambda_r}10^{-s} \tag{32}$$

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Абстракт—Целью работы является рассмотрение двух основных вопросов, в анализе конечного элемента оболочек, а именно скоростей сходимости результатов, полученных из разных схем интерполяции для перемещений в плоскости и нормальных. Далее, исследуется зависимость числа условий полученной алгебраической системы от разных параметров как оболочки так и дискретизации.